A class of infinite dimensional stochastic processes with unbounded diffusion

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Abstract This papers studies a class of Dirichlet forms, with the Wiener trajectories as state space. The diffusion coefficient is constant relative to the state space, it is an unbounded operator in the Cameron-Martin space $\mathbb{H}$. The form is introduced on the space of cylindrical functions with polynomial growth. Necessary and sufficient conditions are presented for the form to be closable. It is shown that under a class of changes of the reference measure, quasi-regularity of the form is retained and local first and second moments of the associated process exist. These local moments are presented in terms of the spectral resolution of the form and the density function corresponding to the change of measure.

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1 Introduction

This paper is concerned with Dirichlet forms of type

$$\mathcal{E}(F,G) = \int \langle DF, ADG\rangle_{\mathbb{H}}\, \varphi d\nu = \int \left\langle DF, \sum_{i=1}^{\infty} \lambda_i \langle S_i, DG\rangle_{\mathbb{H}} S_i \right\rangle_{\mathbb{H}} \varphi d\nu, \tag{1.1}$$

where the diffusion operator A is in general unbounded, cf. [3]. We are interested in weight functions φ of the form

$$\varphi(\gamma) = \exp\left\{\int_0^1 \langle b(\gamma_s),\, d\gamma_s\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^1 \|b(\gamma_s)\|^2\, ds\right\}.$$

This choice of weight functions is motivated by the papers [7, 8] by F.-Y. Wang and B. Wu. In this way it is possible to relate and compare the results of the present paper to the

findings there. We present our ideas in terms of infinite dimensional processes on the classical Wiener space using a coordinate representation. This makes the subject comprehensible, in particular, to readers familiar with the finite dimensional theory. The form is studied on the set of smooth cylindrical functions

$$F, G \in Y = \{F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)) : s_j \text{ is a dyadic point}\},$$

where γ is a Wiener trajectory. We do this as well on the more common set of cylindrical functions

$$F, G \in Z = \{F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)) : s_j \in [0,1]\}.$$

Well-definiteness of $\mathcal{E}$ on Y is a consequence of the fact that the sum in (1.1) is finite. Well-definiteness of $\mathcal{E}$ on Z requires the convergence of the sum in (1.1). These two different initial situations result in possibly different closures of $(\mathcal{E}, Y)$ and $(\mathcal{E}, Z)$ on $L^2(\varphi\nu)$. Using the coordinate representation in (1.1) we give conditions for closability. The requirement of $\varphi^{-1} \in L^1(\nu)$ would be sufficient, for example, for closability in the classical case with bounded cylindrical functions and $\lambda_1 = \lambda_2 = \ldots = 1$ on $L^2(\nu)$, see [4]. However, the paper investigates forms of the structure (1.1) with an in general unbounded diffusion operator. We give necessary and sufficient conditions on the increase of $\lambda_1, \lambda_2, \ldots$ that guarantee closability of $(\mathcal{E}, Y)$ and $(\mathcal{E}, Z)$ on $L^2(\varphi\nu)$ in terms of the Schauder functions S_i, $i \in \mathbb{N}$, the coordinate functions in $\mathbb{H}$. Locality, Dirichlet property, and quasi-regularity of the closure $(\mathcal{E}, Z)$ on $L^2(\varphi\nu)$ is then obtained by using methods of [1, 2, 3, 4, 6]. We are also interested in characterizing the associated process in terms of their local first and second moments of the form

$$\lim_{t\to 0} \frac{1}{t} \int (\mathbf{S}_i(\gamma) - \mathbf{S}_i(\tau)) \; P_\tau(X_t \in d\gamma), \tag{1.2}$$

and

$$\lim_{t\to 0} \frac{1}{t} \int (\mathbf{S}_i(\gamma) - \mathbf{S}_i(\tau))^2 \; P_\tau(X_t \in d\gamma), \tag{1.3}$$

where $\mathbf{S}_i$ are certain linear functions on the trajectory space. We represent these local moments in terms of the sequence $\lambda_1, \lambda_2, \ldots$ and the weight function φ. Corresponding to the closability condition of $(\mathcal{E}, Z)$ on $L^2(\varphi\nu)$, we derive a necessary and sufficient condition for the limits (1.2) and (1.3) to exist. This way lets us obtain compatibility with the classical Kolmogorov characterization of finite dimensional diffusion processes.

2 Formal definitions

We study the form on the space $L^2(\varphi\nu) \equiv L^2(\Omega, \varphi\nu)$ where $\Omega := C_0([0,1]; \mathbb{R}^d) := \{f \in C([0,1]; \mathbb{R}^d, f(0) = 0\}$, ν is the Wiener measure on Ω and φ is a density function specified

below. As stated earlier, the form is given by

$$\mathcal{E}(F,G) = \int \langle DF, ADG\rangle_{\mathbb{H}}\, \varphi d\nu, \quad F,G \in D(\mathcal{E}), \tag{2.1}$$

where $\mathbb{H}$ is the Cameron-Martin space, i.e., the space of all absolutely continuous $\mathbb{R}^d$-valued functions f on $[0,1]$, with $f(0)=0$ and equipped with inner product

$$\langle \varphi, \psi\rangle_{\mathbb{H}} := \int_{[0,1]} \langle \varphi'(x), \psi'(x)\rangle_{\mathbb{R}^d}\, dx.$$

Motivated by [7], we suppose in sections 6 and 7 that $\varphi : \Omega \to [0,\infty]$ has the form

$$\varphi(\gamma) = \exp\left\{\int_0^1 \langle b(\gamma_s),\, d\gamma_s\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^1 \|b(\gamma_s)\|^2_{\mathbb{R}^d}\, ds\right\} \tag{2.2}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is the gradient of a function $f \in C_b^2(\mathbb{R}^d;\mathbb{R})$. Then by Itô's formula, φ defined by (2.2) is both bounded from below and from above, on Ω. We define the set of all cylindrical functions

$$\begin{aligned} Z := \Big\{ & F(\gamma) = f\left(\gamma(s_1),\dots,\gamma(s_k)\right), \gamma \in \Omega : \\ & 0 < s_1 < \dots < s_k = 1, f \in C_p^\infty(\mathbb{R}^{dk}), k \in \mathbb{N}\Big\} \end{aligned}$$

where C_p^∞ denotes smooth functions with polynomial growth. We also define

$$\begin{aligned} Y := \Big\{ & F(\gamma) = f\left(\gamma(s_1),\dots,\gamma(s_k)\right), \gamma \in \Omega : \\ & F \in Z, s_1,\dots,s_k \in \left\{\tfrac{l}{2^n} : l \in \{1,\dots,2^n\}\right\}, n \in \mathbb{N}\Big\}. \end{aligned}$$

For $F \in Z$ and $\gamma \in \Omega$ the gradient operator D is defined by

$$D_s F(\gamma) = \sum_{i=1}^k (s_i \wedge s)(\nabla_{s_i} f)(\gamma), \quad s \in [0,1], \tag{2.3}$$

where $(\nabla_{s_i} f)(\gamma) = (\nabla_{s_i} f)(\gamma(s_1),\dots,\gamma(s_k))$ denotes the gradient of the function f relative to the ith variable while holding the other variables fixed.

We let $(e_j)_{j=1,\dots,d}$ denote the standard basis in $\mathbb{R}^d$ and

$$H_1(t) = 1, \quad t \in [0,1],$$

$$H_{2^m+k}(t) = \begin{cases} 2^{m/2} & \text{if } t \in \left[\frac{k-1}{2^m}, \frac{2k-1}{2^{m+1}}\right) \\ -2^{m/2} & \text{if } t \in \left[\frac{2k-1}{2^{m+1}}, \frac{k}{2^m}\right) \\ 0 & \text{otherwise} \end{cases} \qquad k = 1,\dots,2^m,\ m = 0,1,\dots,$$

denote the system of the Haar functions on $[0,1]$. We also define

$$g_{d(r-1)+j} := H_r \cdot e_j, \quad r \in \mathbb{N},\ j \in \{1,\dots,d\}, \tag{2.4}$$

and

$$S_n(s) := \int_0^s g_n(u)\, du, \quad s \in [0,1],\ n \in \mathbb{N}.$$

3 Definition of the form

We use the following definitions from [3]. We choose a non-decreasing sequence of positive numbers $\lambda_1, \lambda_2, \dots$.

$$D(A) := \left\{ \Phi \in L^2(\Omega \to \mathbb{H}, \nu) : \int \sum_{i=1}^{\infty} \lambda_i^2 \left\langle S_i, \Phi \right\rangle_{\mathbb{H}}^2 \, d\nu < \infty \right\},$$

$$A\Phi(\gamma) := \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, \Phi(\gamma) \right\rangle_{\mathbb{H}} S_i, \quad \gamma \in \Omega, \quad \Phi \in D(A).$$

We can then conclude that

$$\begin{aligned} \mathcal{E}(F,F) &= \int \left\langle DF, ADF \right\rangle_{\mathbb{H}} \varphi d\nu = \int \left\langle DF, \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, DF \right\rangle_{\mathbb{H}} S_i \right\rangle_{\mathbb{H}} \varphi d\nu \\ &= \sum_{i=1}^{\infty} \lambda_i \int \left\langle S_i, DF \right\rangle_{\mathbb{H}}^2 \varphi d\nu < \infty, \quad F \in Y, \end{aligned} \tag{3.1}$$

is well defined since, for $F \in Y$, this is just a finite sum.

4 Closability

In this section we first prove a general closabilty result for the Malliavin gradient. Using this result, a criterion for the closability of the form is formulated. We note that, under these conditions, the form is local.

Lemma 4.1. *If $0 < \varphi \leq c$ for some $c \in \mathbb{R}_+$, and*

$$\frac{f}{\varphi} \in L^1(\nu), \quad \forall f \in Z, \tag{4.1}$$

then $(\mathscr{D}, Z)$ defined by

$$\mathscr{D}(f,g) := \frac{1}{2} \int \left\langle Df, Dg \right\rangle_{\mathbb{H}} \varphi d\nu, \quad f, g \in Z,$$

is closable on $L^2(\varphi\nu)$. Let L denote the generator of $\mathscr{D}$ and note that L is the Ornstein-Uhlenbeck operator.

Proof. Let $u_n \in Z$, $\psi \in Z$ and

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(\varphi\nu), \quad Du_n \underset{n\to\infty}{\longrightarrow} f \quad \text{in } L^2(\varphi\nu; \mathbb{H}).$$

Then

$$\frac{1}{2}\int \left\langle Du_n, \frac{1}{\varphi}D\psi \right\rangle_{\mathbb{H}} \varphi d\nu = \frac{1}{2}\int \langle Du_n, D\psi\rangle_{\mathbb{H}}\, d\nu$$
$$= \int u_n(-L\psi)\, d\nu$$
$$= \int u_n \frac{1}{\varphi}(-L\psi)\, \varphi d\nu.$$

By assumption we have $1/\varphi \cdot (-L\psi)^2 \in L^1(\nu)$, i.e., $1/\varphi \cdot (-L\psi) \in L^2(\varphi\nu)$. Thus

$$\int u_n \frac{1}{\varphi}(-L\psi)\, \varphi d\nu \underset{n\to\infty}{\longrightarrow} 0.$$

Furthermore it turns out that, under (4.1), $\{\frac{1}{\varphi}D\psi : \psi \in Z\}$ is a dense subset of $L^2(\varphi\nu; \mathbb{H})$. It now follows that $f = 0$. Hence the form is closable. $\square$

Remark 4.2. *The closability condition (4.1) would weaken to $1/\varphi \in L^1(\nu)$ if the initial definition of $\mathscr{D}$ was on all bounded $f, g \in Z$. We get compability with [4], section II.2 a).*

We also formulate the following lemma that serves an important role in proving and formulating the closability conditions of the form.

Lemma 4.3. *(a) If $s_k = \sum_{i=1}^{r} c_i \cdot 2^{-i}$, $i_{p,j} = 2^{p-1}(d + j - 1) + 1 + \sum_{q=0}^{p-1} c_q 2^{p-q-1}$ and $j \in \{1, \dots, d\}$ where $c_1, \dots, c_{r-1} \in \{0, 1\}$, $c_r = 1$, $c_0 = c_{r+1} = 0$ and $r \geq 2$, then*

$$\sum_{i=1}^{\infty} \lambda_i \langle S_i(s_k), e_j\rangle_{\mathbb{R}^d}^2 = \lambda_j s_k^2 + \sum_{p=1}^{r} \lambda_{i_{p,j}} 2^{p-1}\left(c_p 2^{-p} + (-1)^{c_p}\sum_{q=p+1}^{r+1} 2^{-q} c_q\right)^2.$$

(b) The relation

$$\sup_{\substack{c_1,\dots,c_{r-1}\in\{0,1\},\\ c_r=1, c_{r+1}=0, j\in\{1,\dots,d\}, r\geq 2}} \sum_{p=1}^{r} \lambda_{i_{p,j}} 2^{p-1}\left(c_p 2^{-p} + (-1)^{c_p}\sum_{q=p+1}^{r+1} 2^{-q} c_q\right)^2 < \infty \tag{4.2}$$

is equivalent to

$$\sum_{p=1}^{\infty} \frac{\lambda_{i_{p,j}}}{2^p} < \infty \quad \textit{for all } j \in \{1, \dots, d\}. \tag{4.3}$$

Proof. (a) The claim is a straightforward consequence of the definitions of the functions S_i, $i \in \mathbb{N}$, using the facts that $\langle S_i(s_k), e_j\rangle_{\mathbb{R}^d} = 0$ for $i \neq j \mod d$ and $S_i(s_k) = 0$ for $i > d2^r$.
(b) Assuming (4.2). Using $c_1 = c_3 = c_5 = \dots = 1$, $c_2 = c_4 = c_6 = \dots = 0$ we get

$$\sup_{\substack{c_1,\dots,c_{r-1}\in\{0,1\},\\ c_r=1, c_{r+1}=0, r\geq 2}} \sum_{p=1}^{r} \lambda_{i_{p,j}} 2^{p-1}\left(c_p 2^{-p} + (-1)^{c_p}\sum_{q=p+1}^{r+1} 2^{-q} c_q\right)^2$$
$$\geq \sum_{p=1}^{\infty} \lambda_{i_{p,j}} 2^{p-1}\left(2^{-p-1}\right)^2 = \frac{1}{8}\sum_{p=1}^{\infty} \frac{\lambda_{i_{p,j}}}{2^p}.$$

Assuming (4.3) we obtain

$$\sup_{\substack{c_1,\ldots,c_{r-1}\in\{0,1\},\\ c_r=1,c_{r+1}=0,r\geq 2}} \sum_{p=1}^{r} \lambda_{i_{p,j}} 2^{p-1} \left(c_p 2^{-p} + (-1)^{c_p} \sum_{q=p+1}^{r+1} 2^{-q} c_q \right)^2$$
$$\leq \sum_{p=1}^{\infty} \lambda_{i_{p,j}} 2^{p-1} \left(2^{-p}\right)^2 = \frac{1}{2} \sum_{p=1}^{\infty} \frac{\lambda_{i_{p,j}}}{2^p}.$$

□

Proposition 4.4. *Let φ satisfy* (4.1).
(a) The form $(\mathcal{E}, Y)$ is closable in $L^2(\varphi\nu)$. Let $(\mathcal{E}, D_Y(\mathcal{E}))$ denote the closure of $(\mathcal{E}, Y)$ in $L^2(\varphi\nu)$.
(b) If

$$\sum_{p=1}^{\infty} \frac{\lambda_{i_{p,j}}}{2^p} < \infty \quad \text{for all } j \in \{1, \ldots, d\} \tag{4.4}$$

where $i_{p,j}$ is as in Lemma 4.3, then $(\mathcal{E}, Z)$ is closable and we let $(\mathcal{E}, D_Z(\mathcal{E}))$ denote the closure of $(\mathcal{E}, Z)$ on $L^2(\varphi\nu)$. Furthermore $D_Z(\mathcal{E}) = D_Y(\mathcal{E})$ under (4.4).
(c) $Z \subset D_Y(\mathcal{E})$ if and only if (4.4).

Proof. (a) We have

$$\mathcal{E}(F, F) = \sum_{i=1}^{\infty} \lambda_i \int \langle S_i, DF \rangle_{\mathbb{H}}^2 \varphi d\nu = \int \left\langle A^{1/2} DF, A^{1/2} DF \right\rangle_{\mathbb{H}} \varphi d\nu.$$

Suppose $\{F_n\}_{n\geq 1} \in Y$ such that $F_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\varphi\nu)$ and $\mathcal{E}(F_n - F_m, F_n - F_m) \to 0$. Since it follows that $A^{1/2} DF_n$ is Cauchy in $L^2(\Omega \to \mathbb{H}, \varphi\nu)$ we may define

$$\psi := \lim_{n\to\infty} A^{1/2} DF_n.$$

We also define

$$Jh := \sum_{i=1}^{\infty} \lambda_i^{-1/2} \langle S_i, h \rangle_{\mathbb{H}} S_i. \tag{4.5}$$

The operator J is bounded in $L^2(\Omega \to \mathbb{H}, \varphi\nu)$ and it follows

$$DF_n = J A^{1/2} DF_n \underset{n\to\infty}{\longrightarrow} J\psi \text{ in } L^2(\Omega \to \mathbb{H}, \varphi\nu).$$

From Lemma 4.1, it is known that (D, Z) is closable on $L^2(\varphi\nu)$. It follows that $DF_n \underset{n\to\infty}{\longrightarrow} 0$ and thus $J\psi = 0$. Since $\lambda_i > 0$ and $\lambda_i^{-1/2} > 0$, (4.5) gives $\psi = 0$. Now $A^{1/2} DF_n \underset{n\to\infty}{\longrightarrow} 0$ and thus $\mathcal{E}(F_n, F_n) = \int \left\langle A^{1/2} DF_n, A^{1/2} DF_n \right\rangle_{\mathbb{H}} \varphi d\nu \to 0$ as $n \to \infty$.
(b) We show that (4.4) implies $Z \subset D_Y(\mathcal{E})$. Then $Y \subset Z \subset D_Y(\mathcal{E})$. Since $(\mathcal{E}, Y)$ is closable with closure $(\mathcal{E}, D_Y(\mathcal{E}))$, cf. (a), $(\mathcal{E}, Z)$ is then also closable and has $(\mathcal{E}, D_Y(\mathcal{E}))$ as its closure

i.e. $D_Y(\mathcal{E}) = D_Z(\mathcal{E})$.

Let $x^v(p)$ denote the vth coordinate of $p \in \mathbb{R}^d$, $v \in \{1, \dots, d\}$. Let us demonstrate that $F(\gamma) = x^v(\gamma(s)) \in D_Y(\mathcal{E})$ for all $v \in \{1, \dots, d\}$ and all $s \in [0,1]$. Fix $s \in [0,1]$, $v \in \{1, \dots, d\}$, and let $s_k \underset{k\to\infty}{\longrightarrow} s$ where s_k is a sequence of dyadic numbers. Now let

$$F_{v,k}(\gamma) := x^v(\gamma(s_k)) \in Y \subset D_Y(\mathcal{E}), \quad k \in \mathbb{N}.$$

Using Lemma 4.3 we have

$$\begin{aligned}\sum_{i=1}^{\infty} \lambda_i \langle S_i, DF(\gamma)\rangle_{\mathbb{H}}^2 &= \sum_{i=1}^{\infty} \lambda_i \langle S_i(s), e_v\rangle_{\mathbb{R}^d}^2 \le \sup_k \sum_{i=1}^{\infty} \lambda_i \langle S_i(s_k), e_v\rangle_{\mathbb{R}^d}^2 \\ &\le \lambda_v + \frac{1}{2}\sum_{p=1}^{\infty} \frac{\lambda_{i_{p,v}}}{2^p} < \infty \end{aligned} \tag{4.6}$$

where $i_{p,v}$ is constructed as in Lemma 4.3. Furthermore $F_{v,k} \underset{k\to\infty}{\longrightarrow} F$ in $L^2(\varphi\nu)$ and

$$\sum_{i=1}^{\infty} \lambda_i \langle S_i, D(F - F_{v,k})(\gamma)\rangle_{\mathbb{H}}^2 \le \sum_{i=n_k}^{\infty} \lambda_i \langle S_i, DF(\gamma)\rangle_{\mathbb{H}}^2$$

for some sequence $n_k \to \infty$ as $k \to \infty$. This says $F_{v,k} \underset{k\to\infty}{\longrightarrow} F$ in $\mathcal{E}_1$-norm and $F \in D_Y(\mathcal{E})$, where we recall that $\|\cdot\|_{\mathcal{E}_1}^2 = \mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2(\varphi\nu)}^2$. Indeed, the same conclusion applies to an arbitrary $F \in Z$ with $F(\gamma) = f(\gamma(s_1), \dots, \gamma(s_k))$, $s_1, \dots, s_k \in [0,1]$ since because of (2.3) we have

$$\langle S_i, DF(\gamma)\rangle_{\mathbb{H}} = \sum_{j=1}^{d}\sum_{i'=1}^{k} \frac{\partial f}{\partial x_{i',j}} (\gamma(s_1), \dots, \gamma(s_k)) \langle S_i(s_{i'}), e_j\rangle_{\mathbb{R}^d}.$$

We then apply (3.1) and (4.6).

(c) Recalling (b), we still have to show that $Z \subset D_Y(\mathcal{E})$ implies (4.4).

We suppose $Z \subset D_Y(\mathcal{E})$. Therefore $x_v(\gamma(s)) \in D_Y(\mathcal{E})$ for all $v \in \{1, \dots, d\}$ and all $s \in [0,1]$. First we check the expression of Lemma 4.3 for all dyadic points s_k. To get a bound on

$$\sum_{i=1}^{\infty} \lambda_i \langle S_i(s), e_j\rangle_{\mathbb{R}^d}^2,$$

when $s \in [0,1]$ is no longer dyadic, we need (4.4). The statement follows. □

Proposition 4.5. *Let φ satisfy* (4.1). *The form $(\mathcal{E}, D_Z(\mathcal{E}))$ is a Dirichlet form on $L^2(\varphi\nu)$.*

Proof. We use Proposition I.4.10 from [4]. It follows that we must show $\mathcal{E}(1 \wedge F^+, 1 \wedge F^+) \le \mathcal{E}(F,F)$. We know that for $F \in Y$

$$\begin{aligned}\mathcal{E}(F,F) &= \sum_{i=1}^{\infty} \lambda_i \int \langle S_i, DF\rangle_{\mathbb{H}}^2 \, \varphi d\nu = \sum_{i=1}^{\infty} \lambda_i \int (\partial_{S_i}, F)^2 \, \varphi d\nu \\ &= \sum_{i=1}^{\infty} \lambda_i \int \left(\frac{d}{dt}\bigg|_{t=0} F(\gamma + tS_i) \right)^2 \varphi d\nu. \end{aligned}$$

Let $\xi_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ be non-decreasing such that $\xi_\varepsilon(t) = t$ for all $t \in [0,1]$, $0 \le \xi'_\varepsilon \le 1$. It follows that $\xi_\varepsilon \circ F \to 1 \wedge F^+$. An application of the chain rule gives

$$\begin{aligned}\mathcal{E}(\xi_\varepsilon \circ F, \xi_\varepsilon \circ F) &= \int \left(\frac{d}{dt}\bigg|_{t=0} \xi_\varepsilon \circ F(\gamma + tS) \right)^2 \varphi d\nu \\ &= \int \xi'_\varepsilon(F(\gamma)) \cdot \left(\frac{d}{dt}\bigg|_{t=0} F(\gamma + tS) \right)^2 \varphi d\nu \\ &\le \int \left(\frac{d}{dt}\bigg|_{t=0} 1 \cdot F(\gamma + tS) \right)^2 \varphi d\nu \\ &= \mathcal{E}(F, F),\end{aligned}$$

from which we derive $\mathcal{E}(1 \wedge F^+, 1 \wedge F^+) \le \mathcal{E}(F,F)$. Thus $\mathcal{E}$ is a Dirichlet form. □

Proposition 4.6. *Let φ satisfy* (4.1)*. The form $(\mathcal{E}, D_Z(\mathcal{E}))$ is local.*

Proof. This is shown in the same manner as in Proposition 3.4 of [3] using the fact that ν-a.e. implies $\varphi\nu$-a.e. □

5 Quasi-regularity

It turns out that the closability condition found in the previous sections is sufficient for the form to be quasi-regular. Throughout this section we assume (4.1).

Proposition 5.1. *Suppose* (4.4)*. The form described by the closure of*

$$\mathcal{E}(F,F) = \sum_{i=1}^{\infty} \lambda_i \int \langle S_i, DF \rangle_{\mathbb{H}}^2 \, d\nu, \quad F \in Z,$$

in $L^2(\varphi\nu)$, is quasi-regular.

Proof. The conditions for closability can be found in Proposition 4.4. We follow the procedure of [3], see also [2, 6]. For $r \in \mathbb{N}$, $k \in 1, \dots, 2^r$, $s_k = k2^{-r}$ let $c_1, \dots, c_r \in \{0,1\}$, such that $s_k = \sum_{i=1}^r c_i 2^{-i}$. Let $x^v(p)$ denote the vth coordinate of $p \in \mathbb{R}^d$, $v \in \{1, \dots, d\}$. Fix $\tau \in \Omega$, $k \in \{1, \dots, 2^r\}$, and $v \in \{1, \dots, d\}$. Consider the functions $f_{v,k,\tau}(p) := x^v(p) - x^v(\tau(s_k)), p \in \mathbb{R}^d$, and

$$F_{v,k,\tau}(\gamma) := f_{v,k,\tau}(\gamma(s_k)) = x^v(\gamma(s_k)) - x^v(\tau(s_k)), \quad \gamma \in \Omega;$$

clearly $F_{v,k,\tau} \in Y$. Using the procedure of Proposition 4.4(b) we get

$$\mathcal{E}(F_{v,k,\tau}, F_{v,k,\tau}) \le C_1 < \infty \tag{5.1}$$

where C_1 the bound obtained from (4.6). Now let

$$G_{n,\tau} = \sup_{\substack{k \in \{1,\dots,n\} \\ v \in \{1,\dots,d\}}} |F_{v,k,\tau}|, \quad n \in \mathbb{N},$$

then

$$\mathcal{E}(G_{n,\tau}, G_{n,\tau}) \le \sup_{\substack{k\in\{1,\dots,n\}\\ v\in\{1,\dots,d\}}} \int \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, DF_{v,k,\tau}(\gamma)\right\rangle_{\mathbb{H}}^2 \, d\nu \le C_1 < \infty,$$

as in [3] Lemma 3.2. We show $G_{n,\tau} \in L^2(\nu)$.

$$\begin{aligned}
E\left[{G_{n,\tau}}^2\right] &\le E\left[\left(\sup_{\substack{v\in\{1,\dots,d\}\\ s\in[0,1]}} \left|x^v\left(\gamma(s)\right) - x^v\left(\tau(s)\right)\right|\right)^2\right] \\
&\le 4E\left[\sup_{\substack{v\in\{1,\dots,d\}\\ s\in[0,1]}} \left(x^v\left(\gamma(s)\right)\right)^2\right] \le 4E\left[\sum_{v=1}^{d} \sup_{s\in[0,1]} \left(x^v\left(\gamma(s)\right)\right)^2\right] \\
&= 4dE\left[\sup_{s\in[0,1]} \left(x^1\left(\gamma(s)\right)\right)^2\right] \le 16d =: C_2 < \infty,
\end{aligned}$$

where the last line is obtained using Doob's inequality. Thus

$$\mathcal{E}_1(G_{n,\tau}, G_{n,\tau}) \le C_1 + C_2 < \infty.$$

Having verified these details the result now follows in the same way as found in [3] Proposition 3.3 *Step 3* and *4*. □

Remark 5.2. *We observe that to obtain* (5.1) *we need the condition* (4.4)*. We recall that by Proposition 4.4(b) we have* $(\mathcal{E}, D_Y(\mathcal{E})) = (\mathcal{E}, D_Z(\mathcal{E}))$*.*

Corollary 5.3. *There exists a diffusion process M associated with* $(\mathcal{E}, D_Z(\mathcal{E}))$*.*

Proof. The result is an immediate consequence of Propositions 4.6 and 5.1 using Theorem IV.3.5 of [4]. □

6 The generator

Let $\varphi(\gamma)$ have the form of (2.2) and $(\mathcal{E}, D_Y(\mathcal{E}))$ given by

$$\mathcal{E}(F,G) = \int \langle DF, ADG\rangle \, \varphi d\nu = \sum_{i=1}^{\infty} \lambda_i \int \partial_{S_i} F \partial_{S_i} G \, \varphi d\nu, \quad F, G \in Y$$

where $D_Y(\mathcal{E})$ is the closure of $(\mathcal{E}, Y)$ on $L^2(\varphi\nu)$. We determine the generator $\mathbf{A}$ of this form i.e.

$$\mathcal{E}(F,G) = \int (-\mathbf{A}F) G \, \varphi d\nu, \quad F \in D(\mathbf{A}), G \in D_Y(\mathcal{E}).$$

Using g_n from (2.4), let $\mathbf{S}_i(\gamma) := \int_0^1 \langle g_i(s), d\gamma_s\rangle_{\mathbb{R}^d}$, $i \in \mathbb{N}$. We note that $\mathbf{S}_1(\gamma), \mathbf{S}_2(\gamma), \dots$ are independent $N(0,1)$.

Proposition 6.1. *We have $Y \subset D(\mathbf{A})$. If $F \in Y$ then*

$$\mathbf{A}F = \sum_{i=1}^{\infty} \lambda_i \left[\partial^2_{S_i} F + \frac{\partial_{S_i}\varphi \partial_{S_i} F}{\varphi} + \mathbf{S}_i \partial_{S_i} F \right]. \tag{6.1}$$

Proof. Let $F, G \in Y$. Once again we note that the following sum is just a finite sum because of the particular structure of Y. We have

$$\begin{aligned}
\mathcal{E}(F,G) &= \sum_{i=1}^{\infty} \lambda_i \int \partial_{S_i} F \partial_{S_i} G \,\varphi d\nu \\
&= \lim_{s\to 0} \sum_{i=1}^{\infty} \lambda_i \frac{1}{s} \left[\int G(\gamma + sS_i) \partial_{S_i} F \varphi d\nu - \int G \partial_{S_i} F \varphi d\nu \right] \\
&= \lim_{s\to 0} \sum_{i=1}^{\infty} \lambda_i \frac{1}{s} \left[\int G \partial_{S_i} F(\gamma - sS_i) \varphi(\gamma - sS_i)\, d\nu(\gamma + sS_i) - \int G \partial_{S_i} F \varphi d\nu \right] \\
&= \lim_{s\to 0} \sum_{i=1}^{\infty} \lambda_i \Bigg[\int G \varphi(\gamma - sS_i) \frac{d\nu \circ T_{sS_i}}{d\nu} \cdot \left(\frac{\partial_{S_i} F(\gamma - sS_i) - \partial_{S_i} F}{s} \right) d\nu \\
&\qquad + \int G \frac{d\nu \circ T_{sS_i}}{d\nu} \partial_{S_i} F \cdot \left(\frac{\varphi(\gamma - sS_i) - \varphi}{s} \right) d\nu \\
&\qquad + \int G \partial_{S_i} F \varphi \left(\frac{\frac{d\nu \circ T_{sS_i}}{d\nu} - 1}{s} \right) d\nu \Bigg] \\
&= \sum_{i=1}^{\infty} \lambda_i \int -G\varphi \partial^2_{S_i} F - G \partial_{S_i} F \partial_{S_i} \varphi + G \varphi \partial_{S_i} F \frac{d}{ds}\bigg|_{s=0} \frac{d\nu \circ T_{sS_i}}{d\nu}\, d\nu,
\end{aligned}$$

where T is the shift operator, $T_{sS_i}(\gamma) = \gamma + sS_i$. Thus the generator satisfies

$$\mathbf{A}F = \sum_{i=1}^{\infty} \lambda_i \left[\partial^2_{S_i} F + \frac{\partial_{S_i}\varphi \,\partial_{S_i} F}{\varphi} - \partial_{S_i} F \frac{d}{ds}\bigg|_{s=0} \frac{d\nu \circ T_{sS_i}}{d\nu} \right]$$

from which we can conclude $F \in D(\mathbf{A})$. The Cameron-Martin formula gives

$$\frac{d}{ds}\bigg|_{s=0} \frac{d\nu \circ T_{sS_i}}{d\nu}(\gamma) = -\mathbf{S}_i(\gamma)$$

and (6.1) follows immediately. ☐

Remark 6.2. *Let b_j denote $\langle b, e_j \rangle_{\mathbb{R}^d}$ and let g_i be as in (2.4). The particular structure of φ gives*

$$\begin{aligned}
\frac{\partial_{S_i}\varphi}{\varphi} &= \sum_{j=1}^{d} \int_0^1 \langle Db_j(\gamma_s), S_i(s) \rangle_{\mathbb{H}}\, d(\gamma_j)_s + \int_0^1 \langle b(\gamma_s), g_i(s) \rangle_{\mathbb{R}^d}\, ds \\
&\quad - \frac{1}{2} \int_0^1 \langle D \|b(\gamma_s)\|^2_{\mathbb{R}^d}, S_i \rangle_{\mathbb{H}}\, ds
\end{aligned}$$

and therefore

$$\mathbf{A}F = \sum_{i=1}^{\infty} \lambda_i \Bigg[\partial_{S_i}^2 F + \mathbf{S}_i \partial_{S_i} F + \Bigg(\sum_{j=1}^{d} \int_0^1 \langle \nabla b_j(\gamma_s), S_i(s) \rangle_{\mathbb{R}^d} \, d(\gamma_j)_s$$

$$+ \int_0^1 \langle b(\gamma_s), g_i(s) \rangle_{\mathbb{R}^d} \, ds - \int_0^1 \Big\langle \sum_{j=1}^{d} b_j(\gamma_s) \nabla b_j(\gamma_s), S_i(s) \Big\rangle_{\mathbb{R}^d} \, ds \Bigg) \partial_{S_i} F \Bigg].$$

7 Local first moment

Let $M = \big((X_t)_{t\in[0,1]}, (P_\gamma)_{\gamma\in\Omega}\big)$ be the corresponding diffusion process defined in Corollary 5.3, S_i^v denote $\langle S_i, e_v \rangle_{\mathbb{R}^d}$, and $\mathbf{S}_i(\gamma)$ be as in Chapter 6. We recall that $(\mathbf{A}, D(\mathbf{A}))$ is the closure of $(\mathbf{A}, Y)$ in the graph norm $\Big(\| \cdot \|_{L^2(\varphi\nu)}^2 + \| \mathbf{A} \cdot \|_{L^2(\varphi\nu)}^2 \Big)^{1/2}$. Thus $\mathbf{S}_i \in Y \subset D(\mathbf{A})$ by definition.

Proposition 7.1. *We have*

$$\lim_{t\to 0} \frac{1}{t} \int (\mathbf{S}_i(\gamma) - \mathbf{S}_i(\tau)) \; P_\tau(X_t \in \, d\gamma) = \lambda_i \left[\frac{\partial_{S_i} \varphi(\tau)}{\varphi(\tau)} + \mathbf{S}_i(\tau) \right].$$

Proof. According to Proposition (6.1), we have $\mathbf{S}_i(\gamma) \in Y \subset D(\mathbf{A})$. Therefore

$$\lim_{t\to 0} \frac{1}{t} \int (\mathbf{S}_i(\gamma) - \mathbf{S}_i(\tau)) \; P_\tau(X_t \in \, d\gamma) = \mathbf{A}(\mathbf{S}_i(\tau))$$

and by (6.1)

$$\begin{aligned} \mathbf{A}(\mathbf{S}_i(\tau)) &= \sum_{j=1}^{\infty} \lambda_j \left[\partial_{S_j}^2 \mathbf{S}_i(\tau) + \frac{\partial_{S_j} \varphi \partial_{S_j} \mathbf{S}_i(\tau)}{\varphi} + \mathbf{S}_j(\tau) \partial_{S_j} \mathbf{S}_i(\tau) \right] \\ &= \lambda_i \left[\frac{\partial_{S_i} \varphi(\tau)}{\varphi(\tau)} + \mathbf{S}_i(\tau) \right]. \end{aligned}$$

$\square$

For a dyadic number, $s_k = \sum_{i=1}^{r} c_i \cdot 2^{-i}$, we note that there exist $N(s_k) \in \mathbb{N}$ such that $\langle (s_k \wedge \cdot), S_i \rangle_{\mathbb{H}} = 0$ whenever $i > N(s_k)$.

Proposition 7.2. *(a) Let $s_k \in [0,1]$ be a dyadic number. Then $x^v(\gamma(s_k)) \in D(\mathbf{A})$ and*

$$\begin{aligned} \mathbf{A} x^v(\tau(s_k)) &= \sum_{i=1}^{N(s_k)} \lambda_i S_i^v(s) \left[\frac{\partial_{S_i} \varphi(\tau)}{\varphi(\tau)} + \mathbf{S}_i(\tau) \right] \\ &= \lim_{t\to 0} \frac{1}{t} \int \left(x^v \left(\gamma(s_k)\right) - x^v \left(\tau(s_k)\right) \right) \; P_\tau(X_t \in \, d\gamma). \end{aligned}$$

(b) Let $v \in \{1, \ldots, d\}$, and let $i_{p,v}$ be as in Lemma 4.3. Suppose (4.4) and

$$\sum_{p=1}^{\infty} \frac{\lambda_{i_{p,v}}^2}{2^p} < \infty \quad \textit{for all } j \in \{1, \ldots, d\}. \tag{7.1}$$

Then $x^v(\gamma(s)) \in D(\mathbf{A})$ and

$$\begin{aligned}\mathbf{A}x^v(\tau(s_k)) &= \sum_{i=1}^{\infty} \lambda_i S_i^v(s) \left[\frac{\partial_{S_i}\varphi(\tau)}{\varphi(\tau)} + \mathbf{S}_i(\tau)\right] \\ &= \lim_{t\to 0} \frac{1}{t} \int \left(x^v(\gamma(s_k)) - x^v(\tau(s_k))\right) P_\tau(X_t \in d\gamma).\end{aligned}$$

Proof. (a) We note

$$\begin{aligned}x^v(\gamma(s_k)) &= \int_0^1 \chi_{[0,s_k]}(u) d\gamma_u^v = \int_0^1 \sum_{i=1}^{\infty} \langle \chi_{[0,s_k]}, H_i \rangle_{L^2} H_i(u) d\gamma_u^v \\ &= \sum_{i=1}^{\infty} \langle \chi_{[0,s_k]}, H_i \rangle_{L^2} \int_0^1 H_i(u) d\gamma_u^v = \sum_{i=1}^{\infty} \langle (s_k \wedge \cdot) e_v, S_i \rangle_{\mathbb{H}} \mathbf{S}_i(\gamma).\end{aligned}$$

Since s_k is dyadic the sum is finite and we get

$$x^v(\gamma(s_k)) = \sum_{i=1}^{N(s_k)} \langle (s_k \wedge \cdot) e_v, S_i \rangle_{\mathbb{H}} \mathbf{S}_i(\gamma) = \sum_{i=1}^{N(s_k)} S_i^v(s_k) \mathbf{S}_i(\gamma).$$

(a) follows immediately.

(b) As in (a) we have $x^v(\gamma(s)) = \sum_{i=1}^{\infty} S_i^v(s) \mathbf{S}_i(\gamma)$. Therefore $\sum_{i=1}^{\infty} \lambda_i S_i^v(s) \mathbf{S}_i$ converges in $L^2(\nu)$ and thus also in $L^2(\varphi\nu)$ if $\sum_{i=1}^{\infty} \lambda_i^2 S_i^v(s)^2 < \infty$. Also $\sum_{i=1}^{\infty} \lambda_i S_i^v(s) \partial_{S_i}\varphi/\varphi$ converges in $L^2(\varphi\nu)$ if $\sum_{i=1}^{\infty} \lambda_i S_i^v(s) < \infty$. Using Lemma 4.3 and (7.1) the result follows. □

Remark 7.3. *For $f, g \in Y$, or $f, g \in Z$ if (4.4) holds, we have for $g \in D(\mathcal{E}) \cap L^\infty$, where $D(\mathcal{E})$ denotes $D_Y(\mathcal{E})$ or $D_Z(\mathcal{E})$ respectively*

$$\begin{aligned}\int g\Gamma(f,f)\,\varphi d\nu &= -\mathcal{E}(g, f^2) + 2\mathcal{E}(fg, f) \\ &= -\int \sum_{i=1}^{\infty} \lambda_i (\partial_{S_i} f^2)(\partial_{S_i} g)\,\varphi d\nu \\ &\quad + 2\int \sum_{i=1}^{\infty} \lambda_i (\partial_{S_i}(fg))(\partial_{S_i} f)\,\varphi d\nu \\ &= \int g \left(2\sum_{i=1}^{\infty} \lambda_i (\partial_{S_i} f)^2\right) \varphi d\nu\end{aligned}$$

where Γ denotes the carré du champ operator (see [1] p. 17). Thus for $f \in Y \cap L^\infty$, or $f \in Z \cap L^\infty$ if (4.4) holds, we have

$$\Gamma(f,f) = 2\sum_{i=1}^{\infty} \lambda_i (\partial_{S_i} f)^2.$$

Using Proposition 7.1 we obtain

$$\lim_{t\to 0}\frac{1}{t}\int \|\mathbf{S}_i(\gamma)-\mathbf{S}_i(\tau)\|^2_{\mathbb{R}^d}\, P_\tau(X_t\in\, d\gamma)=2\lambda_i$$

which formally coincides with Proposition 5.1 in [3]. The same holds for linear combinations of $\mathbf{S}_i(\gamma)$ *belonging to* $D(\mathcal{E})$*.*

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